Vector space

In the following lectures we study the vector space , subspace ,study the linear Independence , basis and the rank of a matrix .

Definition :

Areal vector space is a set V of elements with two operations $\oplus$ and $\odot$ defined with the following properties .

(a) If X and Y are any elements in V . then $X \oplus Y$ is in V (that is closed under the operation $\oplus$) .

1- $X \oplus Y = Y \oplus X$ for all X ,Y in V .
2- $X \oplus (Y \oplus Z) = (X \oplus Y) \oplus Z$ for all X ,Y,Z in V
3- There is a uniqueelement 0 in V such that $X \oplus 0 = 0 \oplus X = X$for every X in V .
4- For each X in V there exists a unique -X in V such that $X \oplus -X = 0$

(b) If X is any element in V and c is any realnumber then $c \odot X$ is in V .

5-$c \odot (X \oplus Y) = c \odot X \oplus c \odot Y$ for any X, Y in V , and any real number c.

6-$(c+d) \odot X = c \odot X \oplus d \odot Y$ for any X in v and any real numbers c and d .

7-$c \odot (d \odot X) = (cd) \odot X$ for any X in v and real numbers c and d .

8-1 $\oplus$ X = X for any X in v .

(V, $\oplus$, $\odot$) is vector space .The operation $\oplus$ is called vector addition .

The operation $\odot$ is called scalar multiplication .
The vector 0 is called Zero vector .

Example 1:
Let R^n be the set of ordered n- tuples (a_1 , a_2 , ……. , a_n) where we define $\oplus$ by (a_1 , a_2 , ……. , a_n)$\oplus$(b_1 , b_2 , ……. , b_n)

=(a_1+ b_1 , a_2+ b_2 , ……. , a_n +b_n) and $\odot$ by c$\odot$(a_1 , a_2 , …. , a_n)
=(ca_1 , ca_2 , ……. , ca_n)
R^n is a vector space .

Example 2:
Let V be the set of ordered triples of real number (a_1 , a_2 , 0) where we define $\oplus$ by (a_1 , a_2 ,0)$\oplus$(b_1 , b_2 , 0)

=(a_1+ b_1 , a_2+ b_2 ,0) and $\odot$ by c$\odot$(a_1 , a_2 ,0)= (ca_1 , ca_2 ,0)
V is a vector space .

Example 3:
Let V be the set of ordered triples of real number (x , y , z) where we define $\oplus$ by (x , y ,z)$\oplus$(x$'$, y$'$,z$'$)

=(x+x$'$, y+ y$'$,z+z$'$)) and $\odot$ by c$\odot$(x , y ,z)= (cx , y ,z)

V is not vector space the property (c+d) $\odot$X = c$\odot$X$\oplus$ d$\odot$X fails to hold

thus (c+d) $\odot$(x , y ,z)=((c+d)x , y ,z),

On other hand c$\odot$(x , y ,z)$\oplus$d$\odot$(x , y ,z)=(cx , y ,z)$\oplus$(dx , y ,z)
=(cx+dx , y+y ,z+z)=((c+d)x ,2y ,2z).

Example 3:
Let V be the set of 2×3 matrices under usual operation of matrix addition and scalar multiplication
V is vector space <u>c.h.</u>

Example 4:
Let V be the set of all real –valued function on R .if f and g are in V we define $f \oplus g$ by $(f \oplus g)(t)= f(t) \oplus g(t)$ and if f and c is a scalar define $c\odot f$ by $c\odot f = c f(t)$.
V is vector space c.h.

Example 5:
Let p_n be the set of all real polynomials of degree $\leq n$ with zero polynomial .if $p(t)=a_0t^n + a_1t^{n-1} + \ldots\ldots + a_{n-1}t + a_n$ and $q(t)=b_0t^n + b_1t^{n-1} + \ldots\ldots + b_{n-1}t + b_n$ are in V we define $p(t) \oplus q(t)$ by $p(t) \oplus q(t) = (a_0+b_0)t^n + (a_1+b_1)t^{n-1} + \ldots\ldots + (a_{n-1}+b_{n-1})t + (a_n+b_n)$ and if c is a scalar define $c\odot p(t)$ by

$c\odot p(t) = (ca_0)t^n + (ca_1)t^{n-1} + \ldots\ldots + ca_{n-1}t + ca_n$

the above definition show that the degree of $p(t) \oplus q(t)$ and $c\odot p(t) \leq n$

$-p(t) = -a_0t^n - a_1t^{n-1} + \ldots\ldots - a_{n-1}t - a_n$ is negative of $p(t)$ and since $a_i+b_i=b_i+a_i$ then $p(t) \oplus q(t) = q(t) + p(t)$
And
$(c+d)\odot p(t) = (c+d)a_0t^n + (c+d)a_1t^{n-1} + \ldots\ldots + (c+d)a_{n-1}t + (c+d)a_n$
$= c(a_0t^n + a_1t^{n-1} + \ldots\ldots + a_{n-1}t + a_n) + d(a_0t^n + a_1t^{n-1} + \ldots\ldots + a_{n-1}t + a_n)$
$= c\odot p(t) \oplus d\odot p(t)$
V is vector space c.h.

Theorem :
If V is a vector space then .

1- $0\odot X = 0$ for any vector X in V

2- $c\odot 0 = 0$ for any scalar c

3- If $c\odot X = 0$ then either c =0 or X = 0

4- $(-1)\odot X = -X$ for any X in V.

Proof:
1) 0X =(0+0)X=0X+0X by (6) of def. adding -0X
0=0X+(-0X)=(0X+0X)+(-0X)
=0X+[0X+(-0X)]

Example 4:

Let V be the set of all real –valued function on R .if f and g are in V we define $f \oplus g$ by $(f \oplus g)(t) = f(t) \oplus g(t)$ and if f and c is a scalar define $c \odot f$ by $c \odot f = c f(t)$.

V is vector space c.h.

Example 5:

Let p_n be the set of all real polynomials of degree $\le n$ with zero polynomial .if $p(t) = a_0 t^n + a_1 t^{n-1} + \dots\dots + a_{n-1} t + a_n$ and $q(t) = b_0 t^n + b_1 t^{n-1} + \dots\dots + b_{n-1} t + b_n$ are in V we define $p(t) \oplus q(t)$ by $p(t) \oplus q(t) = (a_0 + b_0) t^n + (a_1 + b_1) t^{n-1} + \dots\dots + (a_{n-1} + b_{n-1}) t + (a_n + b_n)$ and if c is a scalar define $c \odot p(t)$ by

$c \odot p(t) = (ca_0) t^n + (ca_1) t^{n-1} + \dots\dots + ca_{n-1} t + ca_n$

the above definition show that the degree of $p(t) \oplus q(t)$ and $c \odot p(t) \le n$

$-p(t) = -a_0 t^n - a_1 t^{n-1} + \dots\dots - a_{n-1} t - a_n$ is negative of $p(t)$ and since $a_i + b_i = b_i + a_i$ then $p(t) \oplus q(t) = q(t) + p(t)$

And

$(c + d) \odot p(t) = (c + d) a_0 t^n + (c + d) a_1 t^{n-1} + \dots\dots + (c + d) a_{n-1} t + (c + d) a_n$

$= c(a_0 t^n + a_1 t^{n-1} + \dots\dots + a_{n-1} t + a_n) + d(a_0 t^n + a_1 t^{n-1} + \dots\dots + a_{n-1} t + a_n)$

$= c \odot p(t) \oplus d \odot p(t)$

V is vector space c.h.

Theorem :

If V is a vector space then .

1- $0 \odot X = 0$ for any vector X in V

2- $c \odot 0 = 0$ for any scalar c

3- If $c \odot X = 0$ then either $c = 0$ or $X = 0$

4- $(-1) \odot X = -X$ for any X in V.

Proof:

1) $0X = (0+0)X = 0X + 0X$ by (6) of def. adding $-0X$

$0 = 0X + (-0X) = (0X + 0X) + (-0X)$

$= 0X + [0X + (-0X)]$

$=0X+0=0X.$

2) $c.0 = c.(0+0) = c.0+c.0$

$c.0-c.0=c.0+c.0-c.0$

$0 = c.0$

3) suppose $cX=0$ and $c \neq 0$ then

$$0=(\frac{1}{c}).0=(\frac{1}{c})(cx)=[(\frac{1}{c})c]X=1.X$$

4) $(-1)X+X=(-1)X+(1)X=(-1+1)X=0X=0$ so that $(-1)X=-X$

Definition :

Let V be a vector space and W a nonempty subset of V if W is a vector space with respect to the same operations as these in V , then W is called **a subspace** of V .

Example : If $(V,\oplus,\odot)$ is vector space then

$\{0\}\subseteq V$, $V\subseteq V$ are two subspaces.

Example :

Let W be the set of ordered triples of real number $(a_1, a_2, 0)$ where we define $\oplus$ by $(a_1, a_2, 0)\oplus(b_1, b_2, 0)$

$=(a_1+b_1, a_2+b_2, 0)$ and $\odot$ by $c\odot(a_1, a_2, 0)=(ca_1, ca_2, 0)$

Then $(W,\oplus,\odot)$ is subspace of $(R^3,\oplus,\odot)$.

Theorem:

Let $(V,\oplus,\odot)$ be a vector space and let W be a nonempty subset of V. W is a subspace of V if and only if the following condition hold

1- If X , Y are any vectors in W then $X\oplus Y$ is in W

2- If c is any real number and X is any vector in W then $c\odot X$ is in W .

Example:

Let W be the set of all 2×3 matrices of form

W= $\left\{\begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix}, a,b,c \in R\right\}$,W is subset of vector space V of all 2×3 matrices under usual operations of matrices addition and scalar multiplication then W is subspace of V.

Solution :

Consider $X = \begin{bmatrix} a_1 & b_1 & 0 \\ 0 & c_1 & d_1 \end{bmatrix}$ and $Y = \begin{bmatrix} a_2 & b_2 & 0 \\ 0 & c_2 & d_2 \end{bmatrix}$ in W then

$X+Y = \begin{bmatrix} a_1+a_2 & b_1+b_2 & 0 \\ 0 & c_1+c_2 & d_1+d_2 \end{bmatrix}$ is in W also let $r \in R$

$rX = \begin{bmatrix} ra_1 & rb_1 & 0 \\ 0 & rc_1 & rd_1 \end{bmatrix}$ is in W ,W is subspace of V.

Example:

Let W be the sub set of(R^3 ,⊕,⊙).

W is ordered triples of real number (a , b , 1),

let X=(a_1 , a_2 , 1),Y=(b_1 , b_2 , 1)

X+Y =(a_1+b_1 , a_2+b_2 ,2) Then W is not subspace of (R^3 ,⊕,⊙).

Example 5:

Let W be the set of all real polynomials of degree exactly=2

W is subset of p_2 but not subspace of p_2 since

$2t^2+3t+1$ and $-2t^2+t+2$ is polynomial of degree 1 is not in W.

Exercises:

1- Let W $=\left\{\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}, a=2c+1\right\}$ W is subset of vector space V of all 2×3 matrices under usual operations of matrices addition and scalar multiplication is W is subspace of V.

2- Let W $=\{(a,b,c), b=2a+1\}$ subset of vector space R^3 is W is subspace?

Definition(1 – 7)

Let $X_1, X_2, \ldots\ldots, X_n$ be vectors in a vectors space V. A vector X in V is called linear combination of this vectorsif it can written as X $= c_1X_1+c_2X_2, \ldots\ldots+c_nX_n$ for some real number where $c_1, c_2 \ldots\ldots c_n$ are scalers.

Example: Consider the vector space R^4. let $X_1=(1,2,1,-1)$, $X_2=(1,0,2,-3)$, $X_3=(1,1,0,-2)$ the vector $X=(2,1,5,-5)$ is linear combination of X_1, X_2, X_3 if we find c_1, c_2, c_3 s.t..

$$X = c_1X_1+c_2X_2+c_3X_3$$

$(2,1,5,-5)=c_1(1,2,1,-1)+c_2(1,0,2,-3)+c_3(1,1,0,-2)$

$(2,1,5,-5)=(c_1,2c_1,c_1,-c_1)+ (c_2,0,2c_2,-3c_2)+(c_3,c_3,0,-2c_3)$

$c_1+ c_2+c_3= 2$

$2c_1+c_3 = 1$

$c_1+ 2c_2 \quad = 5$

$-c_1- 3c_2-2c_3= -5$

solving this linear system by Gauss-Jordan we obtain $c_1=1$, $c_2=2$, $c_3= -1$
then X is linear combination of X_1,X_2 , X_3

Example: Consider the vector space R^3 . let $X_1=(1,2,-1)$, $X_2=(1,0,-1)$, is the vector $X=(1,0,2)$ is linear combination of X_1,X_2
if we find c_1 ,c_2 s.t..

$$X = c_1X_1+c_2X_2$$

$(1,0,2) =c_1(1,2,-1)+c_2(1,0,-1)$

$c_1+ c_2= 1$

$2c_1= 0$

$-c_1- 2c_2= 2$

Which has no solution then X is not linear combination of X_1,X_2.

Example: Consider the vector space R^3 . let $X_1=(1,0,1)$, $X_2=(-1,1,0)$, $X_3=(0,0,1)$ is the vector $X=(1,1,1)$ is linear combination of X_1,X_2 , X_3 if we find c_1 ,c_2,c_3 s.t..

$$X = c_1X_1+c_2X_2+c_3X_3$$

$(1,1,1)=c_1(1,0,1)+c_2(-1,1,0)+c_3(0,0,1)$

$c_1- c_2= 1$

$c_2 = 1$

$c_1 + c_2 = a$

$2c_1 \quad = b$

$c_1 + 2c_2 = c$

we obtain

$$\begin{bmatrix} 1 & 1 & : & a \\ 2 & 0 & : & b \\ 1 & 2 & : & c \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & : & 2a-c \\ 0 & 1 & : & c-a \\ 0 & 0 & : & b-4a+2c \end{bmatrix}$$

If $b-4a+2c \neq 0$ then there is no solution to this system hence S does not span P_2.

Linear independence

Definition :Let $S = \{X_1, X_2, \ldots\ldots, X_n\}$ be the set of vectors in a vectors space V. thenS is said to be linearly dependent if there exist constants $c_1, c_2 \ldots\ldots c_n$ not all zero , such that

$c_1X_1+c_2X_2, \ldots\ldots+c_nX_n=0$,other wise . S is called linearly independent

That is S is linearly independent if the equation

$c_1X_1+c_2X_2, \ldots\ldots+c_nX_n=0$ hold only if $c_1=c_2=\ldots\ldots=c_n=0$:

Example: Consider the vector space R^4 . let $X_1=(1,0,1,2)$, $X_2=(0,1,1,2)$, $X_3=(1,1,1,3)$is $S = \{X_1, X_2, X_3\}$ is linearly independent

S0l;

Let $c_1X_1+c_2X_2+c_3X_3=0$ where $c_1, c_2, c_3 \in R$

$c_1(1,0,1,2), +c_2(0,1,1,2)+c_3(1,1,1,3)=(0,0,0,0)$

$(c_1,0,c_1,2c_1), + (0,c_2,c_2,2c_2)+(c_3,c_3,c_3,3c_3)=(0,0,0,0)$

$c_1+ c_3= 0$

$c_2+c_3 = 0$

$c_1+ c_2 +c_3 = 0$

$2c_1+ 2c_2+3c_3= 0$

we obtain $c_1=0$, $c_2=0, c_3= 0$ then S is linearly independent.

Example :

Let V be the vector space R^3 . let$S=\{X_1,X_2 , X_3,X_4 \}$ set of vectors where $X_1=(1,2,-1)$, $X_2=(1,-2,1)$, $X_3=(-3,2,-1)$, $X_4=(2,0,0)$ is the set S linearly independent ?

SOL.:

Let $c_1X_1+c_2X_2+c_3X_3+c_4X_4=0$

$c_1(1,2,-1)+c_2(1,-2,1)+c_3(-3,2,-1)+c_4(2,0,0)=0$

$c_1+ c_2-3c_3 +2c_4= 0$

$2c_1- 2c_2+2c_3 = 0$

$-c_1+c_2 -c_3 = 0$

There are infinitely many solution like $c_1=1$, $c_2=2, c_3=1, c_4= 0$

, then S is linearlydependent.

Example :

Let V be the vector space R^3 . $S=\{i,j ,k \}$ is linearly independent.

Since

$(0,0,0)=(c_1,0,0)+(0,c_2,0)+(0,0,c_3)$

Then $c_1=0$, $c_2=0, c_3=0$

In fact $E_1,E_2,.........,E_n$ are linearly independent in R^n.

Basis and Dimension

Definition:

A set of vectors $S = \{X_1, X_2, \ldots\ldots, X_n\}$ in a vector space V is . called a basis for V if S spans V and S is linearly independent .

Example :

In R^n the unit vector are

$E_1=(1,0,0,\ldots,0)$, $E_2=(0,1,0,\ldots,0)$, …, $E_n=(0,0,\ldots, 1)$

Form a basis for R^n

Example; Let V be the vector space R^4 .let $S=\{X_1, X_2, X_3, X_4\}$ set of vectors where

$X_1=(1,0,1,0)$, $X_2=(0,1,-1,2)$, $X_3=(0,2,2,1)$, $X_4=(1,0,0,1)$ is the set S basis for V?

SOL.:

Let $\qquad c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 = 0$

$$
\begin{aligned}
c_1 \qquad\qquad + c_4 &= 0\\
c_2 + 2c_3 \qquad &= 0\\
c_1 - c_2 + 2c_3 \qquad &= 0\\
2c_2 + c_3 + c_4 &= 0
\end{aligned}
$$

Only solution $c_1 = c_2 = c_3 = c_4 = 0$

, then S is linearly independent.to show S spans R^4 let X=(a,b,c,d,) be any vector in R^4

let $c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 = X$ we can find a solution for c_1 , c_2 , c_3, c_4 by a ,b, c, d .then S spans R^4

then S is a basis for R^4 .

Example :

The set $S=\{t^n, t^{n-1}, \ldots\ldots, t, 1\}$ spans p_n ,since every polynomials of the form $p(t)=a_0t^n+a_1t^{n-1}+\ldots\ldots+a_{n-1}t+a_n$ which is linear combination of elements in S.

S is linearly independent since

$$c_1t^n+c_2t^{n-1}+\ldots\ldots+c_nt+c_{n+1}=0\ldots\ldots(1)$$

holds for every real number t is root of

$p(t)=c_1t^n+c_2t^{n-1}+\ldots\ldots+c_nt+c_{n+1}$.

but nonzero polynomials have only a finite number of roots that (1) only if $c_1=c_2=\ldots\ldots=c_n=c_{n+1}=0$

then S is a basis for p_n

Example: The set $S=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ is a basis for V of all 2×2 matrices to show S is linearly independent

$$c_1\begin{bmatrix}1&0\\0&0\end{bmatrix}+c_2\begin{bmatrix}0&1\\0&0\end{bmatrix}+c_3\begin{bmatrix}0&0\\1&0\end{bmatrix}+c_4\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

then $\begin{bmatrix}c_1&c_2\\c_3&c_4\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ then $c_1=c_2=c_3=c_4=0$

hence S is linearly independent . to show S spans V

Let $c_1\begin{bmatrix}1&0\\0&0\end{bmatrix}+c_2\begin{bmatrix}0&1\\0&0\end{bmatrix}+c_3\begin{bmatrix}0&0\\1&0\end{bmatrix}+c_4\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}a&b\\c&d\end{bmatrix}$

Then $c_1=a, c_2=b, c_3=c, c_4=d$

then S is a basis for V

Theorem 1:If $S=\{X_1, X_2, \ldots\ldots, X_n\}$ is a basis for a vector space V then every vector in V can be written in one and only one way as a linear combination of the vector in S .

Proof :-

First every vector X in V can be written as a linear combination of the vectors in S because S spans V .

Now let

$X = a_1 X_1 + a_2 X_2 + \dots\dots\dots + a_n X_n$ and

$X = b_1 X_1 + b_2 X_2 + \dots\dots\dots + b_n X_n$ we must show that $a_i = b_i$

for $i = 1, 2, \dots\dots, n$ we have

$0 = (a_1 - b_1) X_1 + (a_2 - b_2) X_2 + \dots\dots + (a_n - b_n) X_n$

Since S is linearly independent, we conclude that

$a_i - b_i = 0$ for $i = 1, 2, \dots\dots\dots, n$.

So that $a_i = b_i$

Theorem 2: Let $S = \{X_1, X_2, \dots\dots, X_n\}$ set of non-Zero vectors and let W=spans S then some subset of S is basis for W.

Proof: Ex.(like example)

Example: Let V be the vector space R^4. let $S=\{X_1, X_2, X_3, X_4\}$ set of vectors where

$X_1 = (1,2,-2,1)$, $X_2 = (-3,0,-4,3)$, $X_3 = (2,1,1,-1)$, $X_4 = (-3,3,-9,6)$ Find a subset of S that is basis for W.

SOL.:

observe that every vector X in W is of the form

$aX_1 + bX_2 + c X_3 + d X_4$ ....(1)

to find a basis for W we first determine the set $S=\{X_1, X_2, X_3, X_4\}$ is linearly independent or not .if S linearly independent then S is basis for W. but S is not linearly independent (ch .)

$$X_1 - X_2 - 2 X_3 + 0 X_4 = 0 \dots\dots\dots\dots\dots\dots\dots\dots(2)$$

then

$$X_2 = X_1 - 2 X_3 \qquad \dots\dots\dots\dots\dots\dots\dots(3)$$

Substituting (3) in (1) every vector X in W is of the form

$$(a+b) X_1 + (c - 2b) X_3 + dX_4 \quad \dots\dots\dots(4)$$

Thus W spanned by X_1, X_3, X_4. we check the set $S=\{X_1, X_3, X_4\}$ is linearly independent or not.

We find that X_1, X_2, X_4 is linearly dependent and

$$-3X_1 + 3X_3 + X_4 = 0$$

Then $X_4 = 3X_1 - 3X_3$ ……… (5)

Substituting (5) in (4)) every vector X in W is linear combination of X_1, X_3 then W spanned by X_1, X_3

we check the set

$\{X_1, X_3\}$ is linearly independent or not.

$\{X_1, X_3\}$ is linearly independent and is basis for W .

Linear transformation

Definition :

Let V and W be vector spaces. A linear transformation L of V in to W is a function $L: V \longrightarrow W$

assigning a unique vector L(x) in W to each x in V such that .

a - $L(x + y) = L(x) + L(y)$. for every x and y in V

b- $L(cx) = cL(x)$, for every x in V and every scalar c

Not:

If V=W the linear transformation

$L: V \longrightarrow W$ is also called a linear operator on V.

Example : Let $L: R^3 \longrightarrow R^2$ be defined by

$L(x, y, z) = (x, y)$.

To verify that L is linear transformation we let

$X = (x_1, y_1, z_1)$ and $y = (x_2, y_2, z_2)$

Than $L(x + y) = L((x_1, y_1, z_1) + (x_2, y_2, z_2))$

$= L(x_1+x_2, y_1+y_2, z_1+z_2) = (x_1+x_2, y_1+y_2)$

$= (x_1, y_1) + (x_2, y_2) = L(x) + L(y)$

Also if c is a real number .

Then

$L(cx) = L(cx_1, cy_1, cz_1) = (cx_1, cy_1) = c(x_1, y_1)$

$= c L(x)$

Example : Let $L: R^3 \longrightarrow R^3$ defined by

$L(x, y, z) = (x+1, 2y, z)$. To determine whether L is linear transformation or not

we let $X = (x_1, y_1, z_1)$ and $Y = (x_2, y_2, z_2)$

Than $L(X+Y) = L((x_1, y_1, z_1) + (x_2, y_2, z_2))$

$= L(x_1+x_2, y_1+y_2, z_1+z_2)$

$= ((x_1+x_2)+1, 2(y_1+y_2), z_1+z_2)$

On other hand

$L(x) + L(y) == (x_1+1, 2y_1, z_1) + (x_2+1, 2y_2, z_2)$

$= ((x_1+x_2)+2, 2(y_1+y_2), z_1+z_2)$

Thus $L(x+y) \neq L(x) + L(y)$ L is not linear transformation

Example : Let $L: R^3 \longrightarrow R^3$ be defined by

$L(x) = rx$,r is real number To determine whether L is linear transformation or not

we let $X = (x_1, y_1, z_1)$ and $y = (x_2, y_2, z_2)$

Than $L(X+Y) = L((x_1, y_1, z_1) + (x_2, y_2, z_2))$

$= L(x_1+x_2, y_1+y_2, z_1+z_2)$

$= (r(x_1+x_2), r(y_1+y_2), r(z_1+z_2))$

$= (rx_1, ry_1, rz_1) + (rx_2, ry_2, rz_2)$

$= rL(x) + rL(Y)$ L is linear transformation

Example : Let $L: R^2 \longrightarrow R^3$ be defined by

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$ L is linear transformation since

$X = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $Y = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ Than $L(x+y) = L\left(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$

$$L(x) + L(y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

Example: V= C[0,1] set of all real –valued function that are continuous function is vector space

let W=R and $L: V \longrightarrow W$ is

$L(f) = \int_0^1 f(x)dx$, L is linear transformation Ch?

Theorem :

Let $L:V \longrightarrow W$ be linear trans formation then

$L(c_1X_1 + c_2X_2, \ldots\ldots. + c_n X_n) = c_1 L(X_1) + c_2L(X_2), \ldots\ldots. + c_n L(X_n)$

For any vectors $X_1, X_2 \ldots\ldots\ldots., X_n$ and scalars $c_1, c_2, \ldots\ldots., c_n$.

Proof:

$$L(c_1X_1 + c_2X_2, \ldots\ldots. + c_n X_n) = L(c_1 X_1) + L(c_2X_2) \ldots\ldots. + L(c_n X_n)$$
$$= c_1 L(X_1) + c_2L(X_2), \ldots\ldots. + c_n L(X_n)$$

Theorem :

Let $L:V \longrightarrow W$ be linear trans formation then

(i) $L(0v) = 0w$

(ii) $L(X-Y) = L(X) - L(Y)$ for X,Y in V

Proof :-

(i) We have $0v = 0v + 0v$, so $L(0v + 0v)$

$$L(0v) + L(0v) = L(0v) \text{ .if}$$

We subtract $L(0v)$ from both sides we obtain $L(0v) = 0w$

(ii) $L(X-Y) = L(X+(-Y)) = L(X) + L(-Y)$

$$= L(X) - L(Y)$$

Theorem :

Let $L:V \longrightarrow W$ be linear transformation of an n-dimensional vector space V into a vector space W. Also let S= = $\{X_1, X_2, \ldots\ldots., X_n\}$ be a basis for V . if X is any vector in V then L(X) is completely determined by $\{L(X_1), L(X_2), \ldots\ldots, L(X_n)\}$

Proof :-

Since X is in V, we can write $X = c_1X_1 + c_2X_2, \ldots\ldots. + c_n X_n$

Where $c_1, c_2, \ldots\ldots., c_n$ are real number

Then

$$L(c_1X_1 + c_2X_2, \ldots\ldots. + c_n X_n) = L(c_1 X_1) + L(c_2X_2) \ldots\ldots. + L(c_n X_n)$$
$$= c_1 L(X_1) + c_2L(X_2), \ldots\ldots. + c_n L(X_n)$$

Exercises :

Q1) Is L linear transformation where $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)=\begin{bmatrix} x+y \\ y \\ x-z \end{bmatrix}$?

Q2) Let L: $R^2 \longrightarrow R^2$ linear transformation and $L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)=\begin{bmatrix} 2 \\ -3 \end{bmatrix}$

And $L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ What is $L\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right)$? What is $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$?

Q3)Let $P_2 \longrightarrow P_3$ linear transformation and L(1)=1 ,
$L(t)= t^2$, $L(t^2)=t^3+t$
Find $L(2t^2-5t+3)$, $L(at^2+bt+c)$

The Kernel and Range of linear transformation:

Definition :A Linear trans formation $L:V \longrightarrow W$ is said to be one- to -one if for all X_1 , X_2 in V . $X_1 \neq X_2$ implies $L(X_1) \neq L(X_2)$.
An equivalent statement is that L is one –to-one if for all X_1 , X_2 in V , $L(X_1) = L(X_2)$ implies that $X_1 = X_2$.

Example :
Let L :$R^2 \longrightarrow R^2$ be defined by

$$L(x,y) = (x+y, x-y)$$

tG determine whether L is one -one , we let

$$X_1 = (x_1, y_1) \text{ and } X_2 = (x_2, y_2)$$

then if $L(x_1) = L(x_2)$

$$x_1 + y_1 = x_2 + y_2$$
$$x_1 - y_1 = x_2 - y_2$$

adding these equation , we obtain $2x_1 = 2x_2$ or $x_1 = x_2$
which implies that $y_1 = y_2$ Hence $x_1 = x_2$ and L is one –to -one.

Example :Let L :$R^3 \longrightarrow R^2$ be the linear transformation defined by
$L(x, y, z) = (x, y)$
Since $(1,3,3) \neq (1,3,-2)$ but

$L(1,3,3) = L(1,3,-2) = (1,3)$

We conclude that L is not one-to-one .

Definition :

Let $L:V \longrightarrow W$ A linear transformation .The kernel of L denoted by ker (L) . is the subset of V consisting of all vectors X such $L(X)=0$

$\text{Ker } L = \{ X \in V \,/\, L(X) = 0 \}$.

Example :

Let $L:R^3 \longrightarrow R^2$ defined by $L(x, y, z) = (x, y)$

The vector (0,0,2) is in ker L , since $L(0,0,2) = (0,0)$

However the vector (2,-3,9) is not ker L , since

$L(2,-3,9)=(2,-3)$ to find kerL,we must determine all X in R^3

So that $L(x) = 0$ that ,

However $L(x) = (x_1,x_2)$ thus $(x_1,x_2) = (0,0)$ So $x_1 = 0$, $x_2 = 0$

and x_3 can be any real number . it is clear that

ker L={(0,0,r) ,r is real number}

Consists of the Z-axis in x,y,z three– dimensional space R^3

Example:

Let $L:R^2 \longrightarrow R^2$ be defined by

$L(x,y) = (x+y, x-y)$

Then ker L Consists of of all vectors x in R^2 such that

$L(x) = 0$ thus we must solve the linear system

$x+y = 0$

$x-y = 0$

for x and y . the only solution is $x = 0$ So ker L = { 0 }

Example:

Let $L:R^4 \longrightarrow R^2$ be defined by

$$L(x,y,z,w) = \begin{bmatrix} x+y \\ z+w \end{bmatrix}$$

Then ker L ={ x in R^2:$L(x) = 0$} ker L Consists of of all vectors in the

form $\begin{bmatrix} r \\ -r \\ s \\ -s \end{bmatrix}$ where r,s any real numbers.

Theorem: If $L:V \longrightarrow W$ is linear trans formation . then Ker L is a subspace of V .

Proof :-
First . observe that Ker L is not an empty set since 0v is in ker L .
Also . let x and y be in Ker L .Then since L is linear transformation .

$L(x+y) = L(x) + L(y) = 0w + 0w = 0w$ So x+y is in Ker L .
Also , if c is a scalar . Then since L is linear transformation

$L(cx) = c\,L(x) = c0w = 0w$,So cx is in Ker L .
hence Ker L is subspace of V

Example :
Let $L : R^2 \longrightarrow R^2$ be defined by $L(X,Y) = (X+Y, X-Y)$
Then Ker L ={0},dim(KerL)=0.

Example :
Let $L:R^3 \longrightarrow R^2$ defined by $L(x, y, z) = (x, y)$

Ker L= { X∈ R^2 / L(X) =0} ={(0,0,r): r∈ R } . ,dim(KerL)=1

Example:
Let $L:R^4 \longrightarrow R^2$ be defined by

$L(x,y,z,w) = \begin{bmatrix} x+y \\ z+w \end{bmatrix}$,The basis for KerL is $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ thus :

dim(KerL)=2.

Theorem: If $L:V \longrightarrow W$ is linear transformation . then L(X) is one –one if and only if Ker L={0_V}.
Proof :-
Let X∈ Ker L then L(X)=0_W also L(0_V)=0_W Thus L(X)= L(0_V)
Since L(X) is one –one , hence X=0_V Then Ker L={0_V}

Example : Let L: $R^2 \longrightarrow R^3$ be defined by

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} x+y \\ x-y \\ 2x+3y \end{bmatrix}$$

L is linear transformation since

If $\begin{pmatrix} x \\ y \end{pmatrix}$ is any vector in R^3 then $\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so that

$$L(X) = L(x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix})$$

$$= x\,L(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) + y\,L(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = x\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Coordinate vectors:

Let $L:V \longrightarrow W$ be n-dimensional vector space V with basis $S = \{X_1, X_2, \ldots\ldots, X_n\}$ if $X = a_1X_1 + a_2X_2 + \ldots\ldots + a_nX_n$ Is any vector in V then the vector

$$[X]_S = \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{bmatrix}$$

in R^n is called the coordinate vector of X with respect to the basis S .The components of $[X]_S$ called the coordinates of X with respect to S

Example : Let $S = \{X_1, X_2, X_3\}$ be basis for R^3 where

$$X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

If $X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ then to find $[X]_S$, we must find c_1, c_2, c_3 such that

$$\begin{bmatrix}1\\2\\3\end{bmatrix} = c_1 X_1 + c_2 X_2 + c_3 X_3 \quad \text{thus} \quad \begin{bmatrix}1 & 0 & 1\\-1 & 1 & -1\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

The solution is $c_1 = 2$, $c_2 = 3$, $c_3 = -1$

Then $[X]_S = \begin{bmatrix}2\\3\\-1\end{bmatrix}$

Example : Let S= $\{X_1, X_2, \ldots\ldots, X_n\}$ be a basis be n-dimensional vector space V then since

$X_j = 1X_1 + 0X_2 + \ldots\ldots + 0X_n$. $[X_j]_S = E_j$

Where $\{E_1, E_2, \ldots\ldots\ldots, E_n\}$ a basis for R^n

Example : Let S= $\{t, 1\}$ be a basis for P_1 if P(t)= 5t-2
Then

$[P(t)]_S = \begin{bmatrix}5\\-2\end{bmatrix}$

If T= $\{t+1, t-1\}$ be a basis for P_1

Then $5t - 2 = \frac{3}{2}(t+1) + \frac{7}{2}(t-1)$

Which implies that $[P(t)]_T = \begin{bmatrix}\frac{3}{2}\\\frac{7}{2}\end{bmatrix}$.

Theorem : Let $L: V \longrightarrow W$ be n-dimensional vector space V into an m-dimensional vector space w($n \neq 0$, $m \neq 0$) and let S= $\{X_1, X_2, \ldots\ldots, X_n\}$ and = $\{Y_1, Y_2, Y_3, \ldots Y_m\}$ be bases for V and W , respectively . then the mxn matrix A whose j th column is the coordinate vector $[X_j]_T$ of $L(X_j)$ with respect to T is associated with L and has the following property :

Eigen values And Eigenvectors

Definition: Let A be an $n \times n$ matrix. The real number λ is is called an **Eigen value** of A if there exists a nonzero vector X in R^n such that

$$AX = \lambda X \quad \ldots(1)$$

Every nonzero vector X satisfying (1) is called an **eigenvectors** of A associated with the **Eigen values** λ.

Note:

If X=0 always satisfies Equation(1),but we insist that an eigenvector **X** be a nonzero vector.

Example 1:if A is identity matrix I_n ,then the only eigenvalue is $\lambda=1$;and every nonzero vector in R^n is an eigenvector of A associated with the eigenvalue $\lambda=1$:

I_n X=1X.

Example 2: Let $A=\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$

Then

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So that $X_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of **A** associated with the eigenvalue $\lambda_1=\frac{1}{2}$

Also, $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

So that $X_2=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue $\lambda_2=\frac{-1}{2}$

Figure 5.1 shows that X_1 and AX_1 are parallel, and X_2 and AX_2 are parallel also. this illustrates the fact that if X is an eigenvector of A, then X and AX are parallel.

In figure 5.2 we show X and AX for the cases $\lambda>1, 0<\lambda<1$,and $\lambda<0$.

ample 3: let $A=\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

Then $A\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}=0\begin{bmatrix}1\\0\end{bmatrix}$

So that $X_1=\begin{bmatrix}1\\0\end{bmatrix}$ is an eigenvector of A associated with the eigenvalue

$\lambda_1=0$.also,

$X_2=\begin{bmatrix}0\\1\end{bmatrix}$ is an eigenvector of A associated with the eigenvalue $\lambda_2=1$

Example 3: points out the fact that although the zero vector, by definition, cannot be an eigenvector, the number zero can be eigenvalue.

Example 4: let $A=\begin{bmatrix}1 & 1\\-2 & 4\end{bmatrix}$

We wish to find the eigenvalue of A and their associated eigenvectors.

Thus we wish to find all real numbers λ and all nonzero vectors $X=\begin{bmatrix}x_1\\x_2\end{bmatrix}$ satisfying(1),that is

$$\begin{bmatrix}1 & 1\\-2 & 4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\lambda\begin{bmatrix}x_1\\x_2\end{bmatrix} \quad ....(2)$$

Equation (2) becomes

$$x_1+x_2=\lambda x_1$$
$$-2x_1+4x_2=\lambda x_2$$

or

$$(\lambda-1)x_1- \quad x_2=0$$
$$2x_1+(\lambda-4)x_2=0$$

The homogeneous system of two equations in two unknowns. the homogeneous system in (3) has nontrivial solution if and only if the determinant of its coefficient matrix is zero: thus if and only if

$$\begin{vmatrix}\lambda-1 & -1\\2 & \lambda-4\end{vmatrix}=0$$

This means that $(\lambda-1)(\lambda-4)+2=0$

Definition: let $A=[a_{ij}]$ be an $n\times n$ matrix. the determinant

$$f(\lambda)=|\lambda I_n - A|=\begin{vmatrix} \lambda-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda-a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda-a_{nn} \end{vmatrix} \quad (2)$$

Is called the **characteristic polynomial** of A. the equation

$$f(\lambda)=|\lambda I_n - A|=0$$

is called the **characteristic equation of A.**

Example 5: let

$$A=\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$$

The characteristic polynomial of A is (verify)

$$f(\lambda)=|\lambda I_3 - A|=\begin{vmatrix} \lambda-1 & -2 & -1 \\ -1 & \lambda-0 & 1 \\ -4 & 4 & \lambda-5 \end{vmatrix}=\lambda^3-6\lambda^2+11\lambda-6$$

Theorem: The eigenvalue of **A** are the real roots of the characteristic polynomial of **A**

Proof:

Let λ be an eigenvalue of **A** with associated eigenvector **X**. then

$$AX=\lambda X$$

Which can be rewritten as

$$AX=(\lambda I_n)X$$

Or

$$(\lambda I_n - A)X=0 \qquad (3)$$

A homogeneous system of n equations in n unknowns. This system has a nontrivial solution if and only if the determinant of its coefficient matrix is zero that $|\lambda I_n - A|=0$

Conversely, if λ is a real root of the characteristic polynomial of **A**, then $|\lambda I_n - A|=0$, so the homogeneous system (3) has nontrivial solution **X**. Hence λ is the eigenvalue of **A**

Thus to find the Eigen values of a given matrix A, we must find the real roots of its characteristic polynomial $f(\lambda)$.