Logic

1.1 Propositions and Truth Values : العبارات والقيم الصحيحة

A proposition is declarative statement which is either true or false, but not both. (Propositions are sometimes called **'statements '**).

Examples: -

1. Triangles have four vertices.

2. $6 + 2 = 4$.

3. $5 < 24$.

The truth **(T)** or falsity **(F)** of a proposition is called **Truth Value**. Proposition 3 has a truth value of true (T), and propositions 1&2 have truth values of false (F).

*Questions & demands are not propositions, since they can not be declared true or false .Thus the following are not propositions:

4. Keep off the cat.

5. Did you go to party?

6. Don't say that.

Sentences 4 – 6 are not propositions and therefore cannot be assigned truth values.

* Propositions are denoted using the letters p, q, r, ... Any of these letters may be used to symbolize specific propositions.

***Compound proposition:**

A compound proposition is statement formed by connecting two or more statement, or by negating a simple proposition.

1 .2 Logical connectives:

1) Negation : (~) النفي

If p is any proposition, the negation of p denoted by ~p (or p or ¬ p). And it's a proposition which is false when p is true, and true when p is false.

- We can summarize this in a table ,

p	~ p
T	F
F	T

2) Conjunction : (And) (Λ) أداة الربط (و)

Let p & q be any two propositions, the compound proposition is called conjunction of p & q. And denoted by (p Λ q) .

The following table gives the truth values of p Λ q:

P	Q	p Λ q
T	T	T
T	F	F
F	T	F
F	F	F

From the table it can be seen that the conjunction p Λ q is true only when p and q are both true. Otherwise the conjunction is false.

3) Disjunction : (or) (V) أداة الربط (أو)

Let p & q be any two propositions, compound proposition is called disjunction of p &q. And it's denoted by (p V q) .

The following table gives the truth value of (p V q) :

P	Q	p V q
T	T	T
T	F	T
F	T	T
F	F	F

From the previous table, one can notice that p V q is true when either or both of it's components are true and it's false otherwise.

4) Conditional Propositions: (→) اذا كان...فان The conditional connective (sometimes Called implication) is denoted by → . And reed as if p then q, for any two propositions p & q .

The following is the truth table for p → q :

P	Q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

*

Notice that " the proposition " if p then q " is false only when p is true and q is false . i .e , a true statement can not imply a false one .

5) Biconditional Propositions :(↔) (if and only if)

The biconditional connective is أداة الربط إذا وفقط إذا denoted by ↔ . and expressed by " if and only if .... Then ... ".

The truth table of p ↔ q is ,

p	q	$p \wedge q$	$\overline{p \wedge q}$	$(p \wedge q) \vee (\overline{p \wedge q})$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

The last column of the truth table contains only the truth value T and hence we can deduce that $(p \wedge q) \vee (\overline{p \wedge q})$ is a Tautology.

3) Show that $(p \wedge \overline{q}) \wedge (\overline{p} \vee q)$ is a contradiction .'

Sol :

p	q	$\overline{p}$	$\overline{q}$	$p \wedge \overline{q}$	$\overline{p} \vee q$	$(p \wedge \overline{q}) \wedge (\overline{p} \vee q)$
T	T	F	F	F	F	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	F	T	F

The last column shows that $(p \wedge \overline{q}) \wedge (\overline{p} \vee q)$ is always false , no matter what the truth values of p & q .

Hence $(p \wedge \overline{q}) \wedge (\overline{p} \vee q)$ is a contradiction .

1.4 Logical Equivalence: التكافؤ المنطقي

Two propositions are said to be <u>logically equivalent</u> if they have the same truth values . Using P and Q to denote (possibly) compound propositions , we write $P = Q$ if P&Q are logically equivalent .

Example :- Show that $\bar{p} \vee \bar{q}$ and $\overline{p \wedge q}$ are logically equivalent . i .e , that $(\bar{p} \vee \bar{q}) = (\overline{p \wedge q})$.

<u>Sol :</u>

P	Q	$\bar{p}$	$\bar{q}$	$\bar{p} \vee \bar{q}$	$p \wedge q$	$\overline{p \wedge q}$
T	T	F	F	F	T	F
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

متكافئة

Comparing the columns for $\bar{p} \vee \bar{q}$ & $\overline{p \wedge q}$ we not that the truth values are the same . Hence , $\bar{p} \vee \bar{q}$ & $\overline{p \wedge q}$ are logically equivalent.

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<u>Sol :</u>

p	Q	$\overline{p}$	$\overline{q}$	$\overline{p} \vee \overline{q}$	$p \wedge q$	$\overline{p \wedge q}$
T	T	F	F	F	T	F
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

متكافئة

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Exercises :

1. Prove that $(p \rightarrow q) \equiv (\bar{p} \vee q)$.
2. Prove that $(p \wedge q)$ and $\overline{(p \rightarrow q)}$ are logically equivalent propositions.
3. Show that the biconditional proposition $p \leftrightarrow q$ is logically equivalent to the conjunction of the two conditional propositions $p \rightarrow q$ and $q \rightarrow p$.

1.5 *The Algebra of propositions:* الجبر القضائي

The following is a lit of some important logical equivalences, all in which can be verified using one of the techniques described in (1.4).

These laws hold for any simple propositions p , q and r .

* **Idempotent laws:** قوانين الجمود

$P \wedge P \equiv P$

$P \vee P \equiv P$

* **Commutative laws:** قوانين الإبدال

$p \wedge q \equiv q \wedge p$

$p \vee q \equiv q \vee p$

* **Associative laws:** قوانين التجميع

$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$.

$(p \vee q) \vee r \equiv p \vee (q \vee r)$.

* **Distributive laws:** قوانين التوزيع

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

P	Q	$p \leftrightarrow q$
[redacted]		
T	F	F
F	T	F
[redacted]		

Note that for $p \leftrightarrow q$ to be true , when p and q must both have the same truth value. i.e., both must be true or both must be false.

Examples: -

A B

1. Construct a truth table for (q V p) Λ (~ p V ~ q) .

P q ~p ~q (pvq) (~p V~q) A∧B

[redacted]						
T	T	F	F	T	F	F
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	F	T	F

2. Construct a truth table for (~ q Λ p) V (~q V ~ p) Λp.

A B

P q ~p ~q P∧~q ~p V~q C AVB ∧P

[redacted]							
T	T	F	F	F	F	F	F
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	F
F	F	T	T	F	T	T	F

[redacted]						
T	T	T	F	F	T	F
T	T	F	F	T	T	T
T	F	T	F	F	F	T
T	F	F	F	T	F	F
F	T	T	T	F	T	F
F	T	F	T	T	T	T
F	F	T	T	F	T	F
F	F	F	T	T	T	T

نلاحظ هنا في المثالين c و d يوجد ثلاث متغيرات وهي r , q , p لذا يكون عدد الاحتمالات ثمانية ، حسب القاعدة :

$$\text{عدد الاحتمالات} = 2^{(\text{عدد المتغيرات})}$$

$2^2 = 4 \rightarrow$ (b) , (a) أربع احتمالات كما في

$2^3 = 4 \rightarrow$ (d) , (c) ثماني احتمالات كما في

Exercises :

1) Draw the truth tables for the proposition :

1. $\sim p \rightarrow q$.   2. $\sim q \wedge p$.   3. $(p \vee q) \rightarrow (p \wedge q)$.

4. $\sim p \leftrightarrow (p \wedge q)$.

2) Given the propositions. p, q & r , construct the truth tables for

:

1. $(p \wedge q) \rightarrow \sim r$.   2. $p \wedge (\sim q \vee r)$.   3. $\sim (p \vee q) \leftrightarrow (r \vee p)$.

3. Construct a truth table for :

a) ~q →p. b) ~p↔~q. c) p →(q∧r). d) (~p V q) ↔~r.

a)

p	q	~q	p → ~q
T	T	F	T
T	F	T	T
F	T	F	T
F	F	T	F

b)

p	q	~p	~q	~p ↔ ~q
T	T	F	F	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

c)

p	q	r	q ∧ r	p → (q ∧ r)
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	T	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

1.3 Tautologies and Contradictions :

Definitions:

تكرار في المعنى

1) A **tautology** is a compound proposition which is true no matter what the truth values of it's simple components.

تناقض

2) A **contradiction** is a compound proposition which is false no matter what the truth values of it's simple components.

* We shall denote a tautology by t and a contradiction by f.

Examples :

1) Show that $p \vee \overline{p}$ is a tautology ?

Sol :

Constructing the truth table for $p \vee \overline{p}$, we have :

p	$\overline{p}$	$p \vee \overline{p}$
T	F	T
F	T	T

Note that $p \vee \overline{p}$ is always true (no matter what proposition is substituted for p) and is therefore a Tautology .

2) Show that $\overline{(p \wedge q)}$ is tautology.

Sol :

The truth table of $(p \wedge q) \vee \overline{(p \wedge q)}$ is given below :

Sets and Subsets

2.1 Sets:

A set is to be thought of as any collection of objects whatsoever . The object can also be anything and they are called elements of the set .
The elements contained in a given set need not have anything in common ((other than the obvious common attribute that they all belong to the given set)) , there is no restriction on the number of elements allowed in a set ; there may be an infinite number , a finite number or even no elements at all .

Examples (1) :

1. A set could be defined to contain Picasso , the Babylon Tower and the number π. This is a finite set .
2. The set containing all the positive, even integers is clearly an infinite set.

Notations :

1. We shall generally use upper – case letters to denote sets and lower – case letters to denote elements.

2. The symbol $\in$ denotes ' **belongs to** ' or ' is an element of '.

b) $\{a,b,c\{a,b,c\}\}$.

c) $\{a,\{b,c\},\{a,b,c\}\}$.

d) $\{\{a,b,c\},\{a,b,c\}\}$.

e) $\{a,\{a\},\{\{a\}\},\{\{\{a\}\}\}\}$.

2.2 Subsets:

The set **A** is said to be a subset of the set **B** , if every element of A is also an element of B , and denoted by $A \subseteq B$. symbolically,

$A \subseteq B$ if $\forall x$ $\{x \in A \rightarrow x \in B\}$. Is true.

*علاقة العنصر بالمجموعة هي $\in$ or $\notin$

* علاقة المجموعة بالمجموعة هي $\subseteq$ or $\not\subset$

Examples :

1. $A = \{1,2,3,5\}$ & $B = \{2,1,3,5\}$

   $\therefore A \subseteq B$

   But,

   If $A = \{1,2,4\}$ & $B = \{2,1,3,5\}$

   $A \not\subset B$

2. Let $X = \{1,\{2,3\}\} \rightarrow \{1\} \subseteq X$ but ,

   $\{2,3\} \not\subset X$, However , $\{2,3\}$ is an element of X , so $\{\{2,3\}\} \subseteq X$.

*** Proper Subset :**

If $A \subseteq B$ but $A \neq B$ then we say A is a proper subset of B and denoted by $A \subset B$.

ملاحظات :

ـ كل مجموعة هي مجموعة جزئية من نفسها . $(B \subseteq B)$.

ـ المجموعة الخالية (ϕ) هي جزئية من كل مجموعة مثلاً($\phi \subseteq A$)

Exercises :

1. State whether each of the following statements is true or false:

a) $2 \in \{1,2,3,4,5\}$.

b) $\{2\} \in \{1,2,3,4,5\}$.

c) $2 \subseteq \{1,2,3,4,5\}$.

d) $\{2\} \subseteq \{1,2,3,4,5\}$.

e) $\phi \subseteq \{\phi, \{\phi\}\}$.

f) $0 \in \phi$.

g) $\{1,2,3,4,5\} = \{5,4,3,2,1\}$.

2. list all the subsets of :

a) $\{a,b\}$. b) $\{a,b,c\}$. c) $\{a\}$.

"Relations"

3.1 Relations :

Let A and B be sets . A relation from A to B (or between A and B) is a subset of the Cartesian product $A \times B$.

Remark : The elements of the relation is an ordered pairs .

* (أي أن عناصر العلاقة عبارة عن أزواج مرتبة) .

* We shall use $a \mathcal{R} b$ to denote " a is related to b ". And $a \not\mathcal{R} b$ to denote $(a,b) \notin R$ or " a is not related to b"

Example (1) :

$A = \{1,2,3\}$, $B = \{1,2,3\}$

$A \times B = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$.

1) Find the elements of R_1 iff $a = b$

$\rightarrow R_1 = \{(1,1),(2,2),(3,3)\} \subseteq A \times B$

2) Find the elements of R_2 iff $a < b$

$\rightarrow R_2 = \{(1,2),(2,3),(2,3)\}$

2. Multiplied by scalar :

If k is scalar and $A_{m\times n} = [a_{ij}]_{m\times n}$, then $kA = [ka_{ij}]_{m\times n}$

Example:-

$$3*\begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}_{2\times 2} = \begin{pmatrix} 3 & 6 \\ 0 & -9 \end{pmatrix}_{2\times 2}$$

<u>Note</u> : The division by scalar is like multiplying by [redacted]

3. Addition of Matrices :

Let $A = [\, a_{ij} \,]_{m\times n}$ and $B = [\, b_{ij} \,]_{m\times n}$ be two matrices , then

$$A+B = [\, a_{ij} \,]_{m\times n} + [\, b_{ij} \,]_{m\times n} = C = [\, a_{ij} + b_{ij}]_{m\times n}$$

Example:-

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -3 \end{pmatrix}_{2\times 3} \qquad B = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}_{2\times 3}$$

then,

$$A+B = \begin{pmatrix} 3 & -2 & 3 \\ 1 & 2 & -1 \end{pmatrix}_{2\times 3}$$

- Thus a $\in$ A means (the element) ***a*** belongs to (the set) A.

  And a $\notin$ A means ***a*** does **not belong to A**.

3. Sets can be defined is various ways :

   a) The simplest is by listing (**Enumerate**) it's elements, for example A = $\{1,2,3,4,5\}$ defines the set consisting of the first five positive integers, the order in which the elements are listed is not important.

   c) The other way has the form A = $\{x : P(x)\}$, which read as " the set of all x such that P(x) is true " . Thus,

   A={***x:x is an integer and 1≤ x ≤5***}

* **Finite Set:** A set is said to be finite if it's consist of exactly (n) elements where (n) is some positive integers , otherwise it's infinite .

Example (2) :

1) A = $\{x : x \geq 5\} \rightarrow$ A = $\{5,6,7,.....\}$ infinite set.

2) B = $\{x : x-1=0\} \rightarrow$ B = $\{1\}$ finite set .

3.3 Properties of Relations :

Let R be a relation on set A. We say that R is:

1. Reflexive: الانعكاس

A relation is said to be reflexive if and only if $a\ R\ a$ for every $a \in A$.

Example : $A = \{1,2,3\}$

$R_1 = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3,)\}$

اي كل عنصر يرتبط مع نفسه

R_1 is reflexive

$R_2 = \{(1,1),(2,3),(2,2),(3,1)\}$

$3 \in A$ but $(3,3) \notin R_2$

$\therefore$ R_2 is not reflexive.

2. Symmetric : متناظرة

A relation is said to be symmetric if and only if $a\,R\,b$ implies $b\,R\,a$ [redacted] $a,b \in A$;

Example : $A = \{a,b,c\}$

$R_1 = \{(a,a),(a,b),(b,a),(c,c)\}$

R_1 is symmetric

•اي أن كل عنصر موجود في R_1 يجب أن يكون عكسه موجود أيضا .

Example :

$A = \{1,2,4\}$

$R_1 = \{(1,1),(2,4),(4,2),(1,2),(2,2),(4,4)\}$

$\because$ $(1,2) \in R$ but $(2,1) \notin R$

$\therefore$ R is not symm.

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Let R be a relation on set A. We say that R is:

1. Reflexive: الانعكاس

A relation is said to be reflexive if and only if ***a*** R ***a*** for every a ∈ A.

Matrices

***Linear Equations : -**

Let a_1 , a_2 , …, a_n , b $\in$ R and X_1 , X_2 , …,X_n are variables (unknowns) then the from :

$$a_1 x_1 + a_2x_2 + \ldots + a_nx_n = b_n \cdot$$

Called linear equation, a_1,a_2,..a_n are coefficients and b is the absolute value.

And the form :

[redacted]

[redacted]

⋮

[redacted]

Called system of linear Equations

4.1 Matrices :

The matrix is a rectangular arrangement form consists of orthogonal rows and columns. The coefficients of the linear system are elements of the matrix,